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On Gauss diagrams of periodic virtual knots

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ABSTRACT

In this paper, we study the Gauss diagrams for periodic virtual knots (Theorem 3.1) and show that the virtual knot corresponding to a periodic Gauss diagram is equivalent to the periodic virtual knot whose factor is the virtual knot corresponding to the factor Gauss diagram (Theorem 3.2). We give formulae for the writhe polynomial and the affine index polynomial of periodic virtual knots by using those of factor knots (Corollary 4.2, Corollary 4.6).

Keywords: Gauss diagram; periodic Gauss diagram; periodic virtual knot; covering graph; affine index polynomial; writhe polynomial.

1. Introduction

A *knot* K is an embedding $K: S^1 \hookrightarrow \mathbb{R}^3$ of S^1 into the 3-space $\mathbb{R}^3$. A *diagram* D is a projection of $K(S^1)$ into the plane $\mathbb{R}^2$, in general position, with under/over information. The *Gauss diagram* of K is the domain S^1 on which the crossing information of D is encrypted by chords with suitable information. A Gauss diagram can be a useful tool to study knots. Every Gauss diagram cannot be realized by classical knots, in general. In fact, the realizations of Gauss diagrams are virtual knots.

A *virtual knot diagram* is a generic immersion of S^1 into the plane $\mathbb{R}^2$ possibly with some encircled crossings without under/over information. A virtual knot K is called a *periodic virtual knot* of order n ($n \geq 2$) if it admits a virtual knot diagram D, called an *n-periodic virtual knot diagram*, in $\mathbb{R}^2$ such that D misses the origin and is invariant under the rotation φ of $\mathbb{R}^2$ about the the origin through $\frac{2\pi}{n}$.

In this paper, we study the Gauss diagrams for periodic virtual knots (Theorem 3.1) and show that the virtual knot corresponding to a periodic Gauss diagram is equivalent to the periodic virtual knot whose factor is the virtual knot corresponding to the factor Gauss diagram (Theorem 3.2). We give formulae for the writhe polynomial and the affine index polynomial of periodic virtual knots by using those of factor knots (Corollary 4.2, Corollary 4.6).

2. Preliminaries

A *link* is a finite disjoint union of knots: $L = K_1 \cup \cdots \cup K_n$. Each knot K_i is called a *component* of the link L. Two links L and L' are *equivalent* (or *isotopic*) if one can be transformed into the other via a deformation of $\mathbb{R}^3$ upon itself. A *diagram* of a link L is a regular projection image $p(L)$ from the link L into $\mathbb{R}^2$ such that the over-path and the under-path at each double points of $p(L)$ are distinguished.

In 1996, Kauffman introduced the notion of a virtual knot. A *virtual knot diagram* is a knot diagram in $\mathbb{R}^2$ possibly with some encircled crossings without over/under information. Such an encircled crossing is called a *virtual crossing*, see Fig. 1 as example.

Two virtual knot diagrams are *equivalent* if one can be transformed into another by a finite sequence of generalized Reidemeister moves in Fig. 2, which consists of classical Reidemeister moves and virtual Reidemeister moves. An equivalence class of a virtual knot diagram is called a *virtual knot*.

From a topological viewpoint, the underlying space of a diagram of a link is a 4-valent graph embedded in S^2 in Fig. 3.

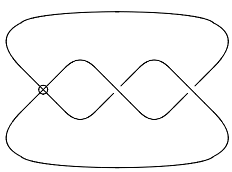

Fig. 1. A virtual trefoil knot diagram.

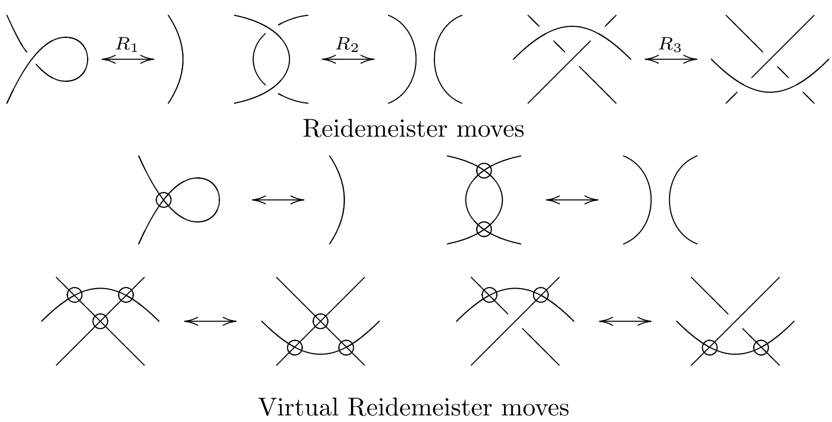

Fig. 2. General Reidemeister moves.

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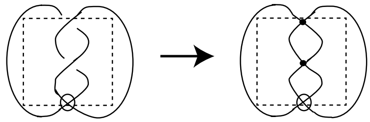

Fig. 3.

Here, we review the results in topological graph theory that can be used to study links, see [1, 5, 10] in detail. A *graph* Γ consists of a finite set $V(\Gamma)$ of vertices and a finite set $E(\Gamma)$ of edges. If the edges of a graph Γ have a direction associated with them, the graph Γ is called a *directed graph*. An embedding of Γ into a surface S is called a 2-*cell embedding* if each component of $S\backslash i(\Gamma)$, called a *region* of the embedding, is homeomorphic to the standard disc. For a vertex $v_i \in V(\Gamma)$, let $V(v_i)$ be the set of all vertices incident to v_i, and let $P_{v_i} : V(v_i) \to V(v_i)$ be a cyclic permutation on $V(v_i)$. We call $(P_{v_1}P_{v_2}\cdots P_{v_n})$ *a rotation scheme* of the embedding. It is well-known that there is a one-to-one correspondence between the set of all embeddings of a graph and the set of all rotation schemes.

Let Γ be a graph and A a finite group. Let $\phi : E(\Gamma) \to A$ be a function, called a *voltage assignment*, satisfying $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\Gamma)$ where e^{-1} means the edge e with the reversed orientation of $E(\Gamma)$. The values of ϕ are called *voltages* and A is called the *voltage group*. We call a triple (Γ, A, ϕ) a *voltage graph*. The *covering graph* $\Gamma \times_\phi A$ for (Γ, A, ϕ) has the vertex set $V(\Gamma) \times A$ and each edge $e = uv$ of Γ determines the edges $(e, g) = (u, g)(v, g\phi(e))$ of $\Gamma \times_\phi A$, for all $g \in A$. Notice that $\Gamma \times_\phi A$ is a $|A|$-fold regular covering space of Γ; in fact, every regular covering space of Γ can be obtained in this manner.

Now consider a voltage graph (Γ, A, ϕ) which is 2-cell embedded in an orientable surface S, as described algebraically by the rotation scheme $P = (P_1, P_2, \ldots, P_n)$. We define the *lift* $\tilde{P}$ of P to $\Gamma \times_\phi A$ as follows: if $P_v(vu) = (vw)$, then

$$\tilde{P}_{(v,g)}((v, g)(u, g\phi(vu))) = (v, g)(w, g\phi(vw)),$$

for each $g \in A$, see Fig. 4. Since $\tilde{P} = \{\tilde{P}_{(v,g)} \,|\, (v, g) \in V(\Gamma \times_\phi A)\}$ is a rotation scheme of $\Gamma \times_\phi A$, it determines the natural embedding of $\Gamma \times_\phi A$ into a surface $\tilde{S}$.

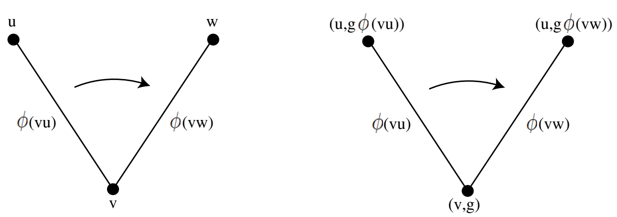

Fig. 4.

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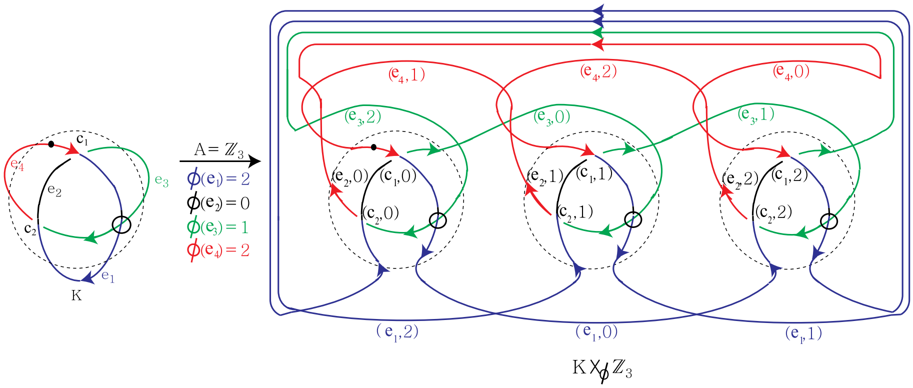

Fig. 5.

Example 2.1. Let a virtual diagram K be given in the left of Fig. 5 and $\phi: E(K) \to \mathbb{Z}_3$ a voltage assignment defined by $\phi(e_1) = 2$, $\phi(e_2) = 0$, $\phi(e_3) = 1$ and $\phi(e_4) = 2$ where e_1, e_2, e_3, $e_4 \in E(K)$. Since K has 2-vertices, c_1 and c_2, 4-edges, e_1, e_2, e_3 and e_4, and since the voltage group $\mathbb{Z}_3$ has three elements 0, 1 and 2, the covering diagram has 6-vertices, $(c_1, 0)$, $(c_2, 0)$, $(c_1, 1)$, $(c_2, 1)$, $(c_1, 2)$ and $(c_2, 2)$, and 12-edges, $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$, $(e_4, 0)$, $(e_1, 1)$, $(e_2, 1)$, $(e_3, 1)$, $(e_4, 1)$, $(e_1, 2)$, $(e_2, 2)$, $(e_3, 2)$ and $(e_4, 2)$. Since $\phi(e_1) = 2$, and the initial vertex of e_1 is c_1 and the terminal vertex is c_2 in the base diagram K, it follows that for $i \in \mathbb{Z}_3$, the edge (e_1, i) of the covering diagram runs from the vertex (c_1, i) to the vertex $(c_2, i+2)$. Since $\phi(e_2) = 0$, and the initial vertex of e_2 is c_2 and the terminal vertex is c_1 in the base diagram K, it follows that for $i \in \mathbb{Z}_3$, the edge (e_2, i) of the covering diagram runs from the vertex (c_2, i) to the vertex (c_1, i). Similarly, we can draw (e_3, i) and (e_4, i) for $i \in \mathbb{Z}_3$. Then we obtain the covering diagram $K \times_\phi \mathbb{Z}_3$ as seen in the right of Fig. 5.

Notice that there are many new born 4-valent vertices in the diagram of the right part in Fig. 5 which are neither classical nor virtual crossings. By considering such vertices as virtual crossings, we get the virtual diagram, on which $\mathbb{Z}_3$ can act, see Fig. 6. Note that the factor $(K \times_\phi \mathbb{Z}_3)_*$ in Fig. 6 is equivalent to the base diagram K in Fig. 5. In fact, every m-periodic virtual link diagram can be constructed in this way.

3. Periodic Gauss Diagrams

We take a short review how to construct a Gauss diagram from a given knot, see [3, 6]. Let K be an oriented virtual diagram with n classical crossings. Fix a point, say the initial point, on the arc in K. Draw a circle with the initial point. Without loss of generality, we assume that the circle is oriented counterclockwise. Go along the arc from the initial point to that point according to the orientation of K. We have a sequence $\{c_1, c_2, \dots, c_n\}$ of labels for the crossings. These sequence

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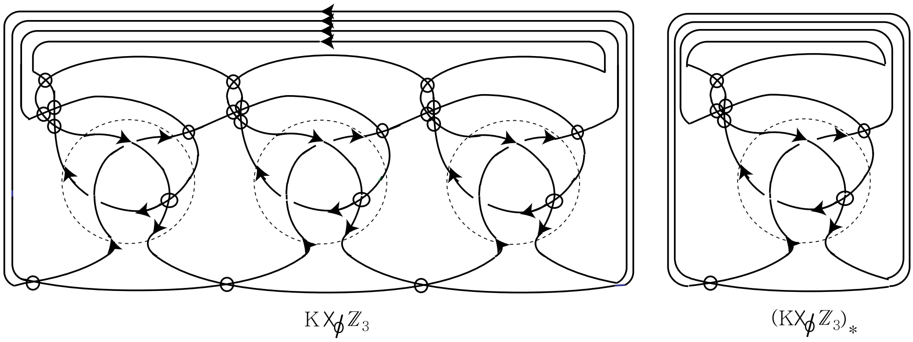

Fig. 6.

is labeled in sequence from the initial point to that point on the circle, each label is repeated twice to indicate a walk along the diagram, we assign the sequence on the circle. That is, the classical crossings $c_1, c_2, \ldots, c_n$ of K correspond to $2n$ vertices $c_1^o, c_1^u, c_2^o, c_2^u, \ldots, c_n^o, c_n^u$ on the circle, where c_i^o and c_i^u correspond to the over-arc and the under-arc of c_i, respectively. If the circle consists of the follows: the chords are connected to the c_i^o and c_i^u, the chords are endowed with arrows from over-arc c_i^o to under-arc c_i^u with the signs of crossings for each i for each $1 \le i \le n$, the result is called a *Gauss diagram of the knot*. See Fig. 7 as an example.

Proposition 3.1 ([4]). *A Gauss diagram defines a virtual knot diagram up to virtual Reidemeister moves.*

Observe that G is the Gauss diagram of a diagram K if and only if whenever e is an edge of K with the initial crossing c_i and the terminal crossing c_j, there is the corresponding edge e' of G on the circle whose ends meet two chords c_i' and

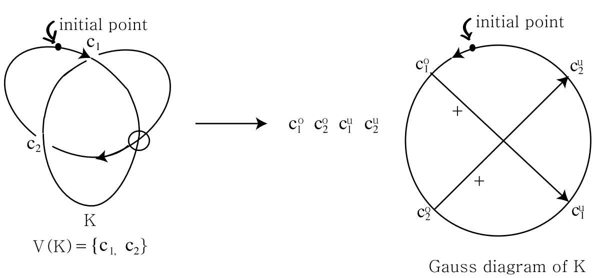

Fig. 7.

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c'_j. The directions and the signs on the chords c'_i and c'_j depend on the crossing information of c_i and c_j, respectively.

Let K be an oriented virtual knot diagram with n vertices $c_1, c_2, \ldots, c_n$ and $G(K)$ the associated Gauss diagram of K. Let $\phi: E(K) \to \mathbb{Z}_m$ be a voltage assignment on $E(K)$. Suppose that the net voltage $\sum_{e \in E(K)} \phi(e)$ is relative prime to m so that $K \times_\phi \mathbb{Z}_m$ is a virtual knot, see [10, Theorem 10-8]. From now on, we will try to find out the Gauss diagram of $K \times_\phi \mathbb{Z}_m$.

Note that a Gauss diagram $G(K)$ can be viewed as a 3-valent graph whose edges consist of the edges on the circle and chords. The edge e' of $G(K)$ on the circle corresponds to the edge e of K. Define a map $\varphi : E(G(K)) \to \mathbb{Z}_m$ by

$$\varphi(e') = \begin{cases} \phi(e), & \text{if } e' \text{ is an edge on the circle of } G(K); \\ 0, & \text{if } e' \text{ is a chord of } G(K). \end{cases}$$

Since φ is a voltage assignment, we have the covering graph $G(K) \times_\varphi \mathbb{Z}_m$. From the construction of the covering graph, we can lift the direction information and the sign information of chords of $G(K)$ to those of $G(K) \times_\varphi \mathbb{Z}_m$, so that the covering $G(K) \times_\varphi \mathbb{Z}_m$ can be a Gauss diagram on which $\mathbb{Z}_m$ can act. We call the Gauss diagram $G(K) \times_\varphi \mathbb{Z}_m$ an *m-periodic Gauss diagram* whose factor Gauss diagram is $G(K)$. See Fig. 8 as an example.

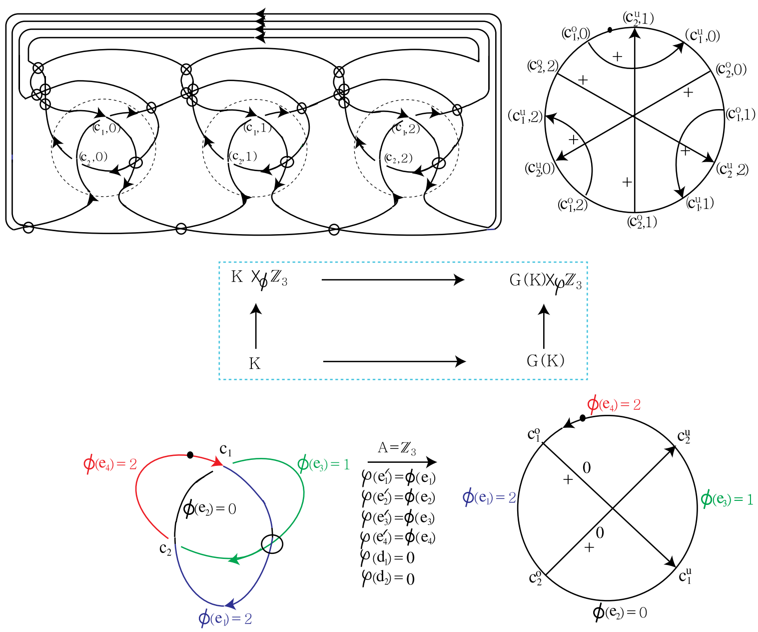

Fig. 8.

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The following is one of the main results.

Theorem 3.1. *The periodic Gauss diagram $G(K) \times_{\varphi} \mathbb{Z}_m$ is the Gauss diagram of the periodic virtual knot $K \times_{\phi} \mathbb{Z}_m$.*

Proof. Let $\widetilde{e}$ be an edge of $K \times_{\phi} \mathbb{Z}_m$. Then there are an edge e in $E(K)$ with the initial vertex c_i and the terminal vertex c_j, and an element g in $\mathbb{Z}_m$ such that $\widetilde{e} = (e, g)$ with the initial vertex (c_i, g) and the terminal vertex $(c_j, g + \phi(e))$. The corresponding edge e' of the Gauss diagram $G(K)$ of K meets two chords c'_i and c'_j, while the direction and the sign on the chords c'_i and c'_j depend on the crossing information of c_i and c_j, respectively.

From the construction of the covering $G(K) \times_{\varphi} \mathbb{Z}_m$, we know that for the edge e' on $G(K)$ and the element g in $\mathbb{Z}_m$, the edge (e', g) meets chords (c'_i, g) and $(c'_j, g + \varphi(e'))$, and the direction and the sign on the chords (c'_i, g) and $(c'_j, g + \varphi(e'))$ coincide with those of chords c'_i and c'_j, respectively.

On the other hand, the edge $(e, g)'$ of the Gauss diagram $G(K \times_{\phi} \mathbb{Z}_m)$ of $K \times_{\phi} \mathbb{Z}_m$ corresponding to the edge (e, g) of $K \times_{\phi} \mathbb{Z}_m$ meets two chords $(c_i, g)'$ and $(c_j, g + \phi(e))'$, and the direction and the sign on the chords $(c_i, g)'$ and $(c_j, g + \phi(e))'$ depend on the crossing information of c_i and c_j, respectively. Because the crossing information of crossings (c_i, g) and $(c_j, g + \phi(e))$ coincides with that of the crossings c_i and c_j, respectively.

If we identify the chord $(c_i, g)'$ and the edge $(e, g)'$ with the chord (c'_i, g) and the edge (e', g), respectively, then $(c_j, g + \phi(e))' = (c'_j, g + \varphi(e'))$ since $\varphi(e') = \phi(e)$. Then the Gauss diagram $G(K) \times_{\varphi} \mathbb{Z}_m$ is the same with the Gauss diagram $G(K \times_{\phi} \mathbb{Z}_m)$ because the direction and the sign of chords of $G(K) \times_{\varphi} \mathbb{Z}_m$ coincide with those of $G(K \times_{\phi} \mathbb{Z}_m)$. □

Example 3.2. The following diagram is useful to understand the proof of the previous theorem.

Now, let G be a Gauss diagram and K_G an associated virtual knot diagram of G. Let $\varphi : E(G) \to \mathbb{Z}_m$ be a voltage assignment. Suppose that the net voltage $\sum_{e \in E(G)} \varphi(e)$ is relative prime to m so that $G \times_{\varphi} \mathbb{Z}_m$ is an m-periodic Gauss diagram of a virtual knot whose factor Gauss diagram is G. From now on, we will try to find out the virtual knot corresponding to the Gauss diagram $G \times_{\varphi} \mathbb{Z}_m$. The edge e of K_G corresponds to the edge e' of G on the circle. Define a map $\phi : E(K_G) \to \mathbb{Z}_m$ by

$$\phi(e) = \varphi(e').$$

Since ϕ is a voltage assignment, we have the covering graph $K_G \times_{\phi} \mathbb{Z}_m$. From the construction of $K_G \times_{\phi} \mathbb{Z}_m$, the covering $K_G \times_{\phi} \mathbb{Z}_m$ is an m-periodic virtual knot diagram whose factor is K_G.

Theorem 3.2. *The virtual knot corresponding to a periodic Gauss diagram $G \times_{\varphi} \mathbb{Z}_m$ is equivalent to the periodic virtual knot $K_G \times_{\phi} \mathbb{Z}_m$ whose factor K_G is the virtual knot corresponding to the factor Gauss diagram G.*

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Proof. By Theorem 3.1, the Gauss diagram $G \times_\varphi \mathbb{Z}_m$ is the same with the Gauss diagram of $K_G \times_\phi \mathbb{Z}_m$. Then by Proposition 3.1, the virtual knot corresponding to $G \times_\varphi \mathbb{Z}_m$ is equivalent to the virtual knot $K_G \times_\phi \mathbb{Z}_m$. $\square$

4. The Affine Index Polynomial and the Writhe Polynomial of a Periodic Virtual Knot

In the previous section, we constructed an m-periodic Gauss diagram $G(K) \times_\varphi \mathbb{Z}_m$ whose factor is $G(K)$, and showed that $G(K) \times_\varphi \mathbb{Z}_m$ is the Gauss diagram of a periodic virtual knot $K \times_\phi \mathbb{Z}_m$. Hence, if there is an invariant of virtual knots which can be determined by Gauss diagrams, then one may calculate the invariants of periodic knots. It is known that the affine index polynomial and the writhe polynomial are such invariants. In this section, we give formulae for the writhe polynomial and the affine index polynomial of periodic virtual knots by using those of factor knots.

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4.1. *The affine index polynomial of a periodic virtual knot*

Kauffman introduced the affine index polynomial of a virtual knot L [7]. In order to define the affine index polynomial of L, we give an integer to each edge in a diagram K of L. Here an edge means a part of K from a classical crossing to the next classical crossing along the orientation of K. First, choose any edge e of K and assign any integer a to e. The terminal point c of e is a classical crossing of K. There are four cases according to under/over information at c and to the orientation of K in Fig. 9. Notice that we do not concern the virtual crossings. To the edge f of K meeting e linearly at c, assign $a+1$ or $a-1$ according to the rule in Fig. 9. It is known that one can assign an integer to each edge of K by repeating this process. The labeling rule around a crossing is given in Fig. 10.

The *weight* of a crossing c_i is defined by

$$W(c_i) = \begin{cases} a-b-1, & \omega(c_i) = +1; \\ b+1-a, & \omega(c_i) = -1. \end{cases}$$

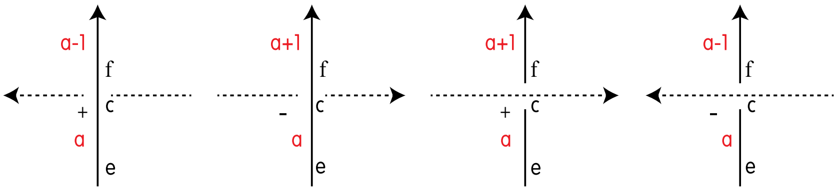

Fig. 9.

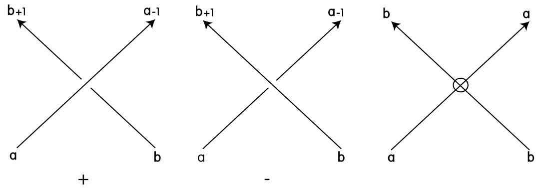

Fig. 10.

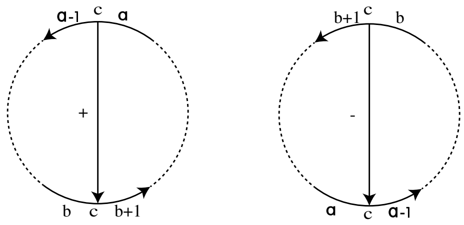

Fig. 11.

The *affine index polynomial* $A_K(t)$ of a virtual knot K is defined by

$$A_K(t) = \sum_{c_i} \omega(c_i)(t^{W(c_i)} - 1),$$

where $\omega(c_i)$ is the writhe number of c_i. The integer labeling is not unique, but the affine index polynomial is well-defined and an invariant of virtual knots, see [7].

Let $G(K)$ be the corresponding Gauss diagram of K. The labeling rule of K around a crossing c is translated as the labeling rule of $G(K)$ around the chord corresponding to the crossing c as in Fig. 11. Notice that our labeling of K or, $G(K)$ is not the voltage assignment for the covering construction.

Let K be a virtual knot diagram and $\phi : E(K) \to \mathbb{Z}_m$ a voltage assignment. Suppose that the net voltage $\sum_{e \in E(K)} \phi(e)$ is relative prime to m so that $K \times_\phi \mathbb{Z}_m$ is an m-periodic knot diagram. Let c be a crossing of K. Then the fiber of c consists of the crossings $(c, 0), (c, 1), \ldots, (c, m-1)$ of $K \times_\phi \mathbb{Z}_m$.

Lemma 4.1. *$\omega(c) = \omega((c, g))$ and $W(c) = W((c, g))$ for all $g \in \mathbb{Z}_m$.*

Proof. From Theorem 3.1, the Gauss diagram $G(K) \times_\varphi \mathbb{Z}_m$ is the Gauss diagram of $K \times_\phi \mathbb{Z}_m$ whose factor is $G(K)$. Let $\widetilde{c'}$ be a chord of $G(K) \times_\varphi \mathbb{Z}_m$. Then there are a chord c' in $G(K)$ and an element k in $\mathbb{Z}_m$ such that $\widetilde{c'} = (c', k)$. Since the sign of the chord (c', k) coincides with that of c', $\omega(c') = \omega((c', k))$. Hence $\omega(c) = \omega((c, g))$ for all $g \in \mathbb{Z}_m$.

In order to calculate the weight $W(c')$ of the chord c', we need a labeling for every edge of $G(K)$. Without loss of generality, we give 0 to the edge e' of $G(K)$ on the circle whose initial vertex is one of two ends of the chord c', say c'^o. Then

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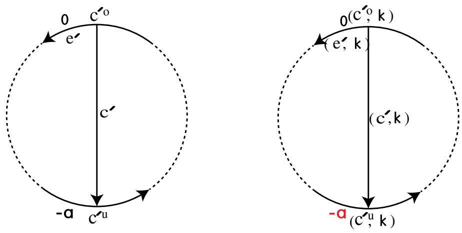

Fig. 12.

the labeling of edges of $G(K)$ is given so that the weight $W(c')$ of the chord c' is calculated by the labeling rule defined as in Fig. 11, say $W(c') = a$. Then by the definition of the weight, the local shape of $G(K)$ near the chord c' is depicted as the left of Fig. 12. Note that since the value of the edge e' is 0, the sum of labeling of all edges of $G(K)$ is 0. To calculate the weight $W((c', k))$ of the chord (c', k), one can give 0 to the edge (e', k) of $G(K) \times_\varphi \mathbb{Z}_m$ whose initial vertex is (c'^o, k). See the right of Fig. 12.

On the other hand, let $V(G(K))$ be the set of all vertices of $G(K)$ and $T(G(K))$ the set of all vertices of $G(K)$ from c'^o to c'^u according to the orientation of the circle of $G(K)$. From the construction of $G(K) \times_\varphi \mathbb{Z}_m$, the set of all vertices from the vertex (c'^o, k) to the vertex (c'^u, k) is $T(G(K)) \cup (\bigcup_{i=0}^{l}(V(G(K))))$ for some $l \in \mathbb{Z}_m$. Since the sum of labeling of all edges of $G(K)$ is 0, the value of the labeling on the adjacent left edge of (c'^u, k) is $-a$ because the sign and direction of chords of $G(K) \times_\varphi \mathbb{Z}_m$ coincide with those of $G(K)$. Then $W((c', k)) = W(c')$. Hence $W(c) = W((c, g))$ for all $g \in \mathbb{Z}_m$. □

Example 4.1. Figure 13 is an illustration of the proof for Lemma 4.1.

Theorem 4.2. *The affine index polynomial $A_{K \times_\phi \mathbb{Z}_m}(t)$ of $K \times_\phi \mathbb{Z}_m$ is given as*

$$A_{K \times_\phi \mathbb{Z}_m}(t) = m \cdot A_K(t).$$

Proof. Suppose that the set $V(K)$ of all classical crossings of K is $\{c_1, c_2, \ldots, c_n\}$. Then the set $V(K \times_\phi \mathbb{Z}_m)$ of all classical crossings of $K \times_\phi \mathbb{Z}_m$ is $\{(c_i, g) \,|\, c_i \in$

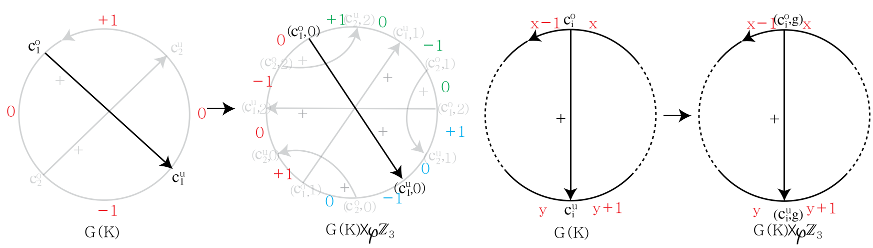

Fig. 13.

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$V(K), g \in \mathbb{Z}_m\}$.

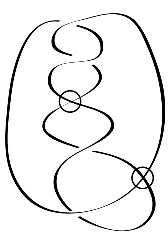
$$\begin{aligned}
A_{K\times_\phi \mathbb{Z}_m}(t) &= \sum_{(c_i,g)} \omega((c_i,g))(t^{W((c_i,g))}-1)\\
&= \sum_{(c_i,g)} \omega((c_i,g))t^{W((c_i,g))} - \sum_{(c_i,g)} \omega((c_i,g))\\
&= \sum_{(c_1,g)} \omega((c_1,g))t^{W((c_1,g))} + \sum_{(c_2,g)} \omega((c_2,g))t^{W((c_2,g))}\\
&\quad + \cdots + \sum_{(c_n,g)} \omega((c_n,g))t^{W((c_n,g))} - \sum_{(c_i,g)} \omega((c_i,g))\\
&= m\sum_{c_1}\omega(c_1)t^{W(c_1)} + m\sum_{c_2}\omega(c_2)t^{W(c_2)}\\
&\quad + \cdots + m\sum_{c_n\in K}\omega(c_n)t^{W(c_n)} - m\sum_{c_i}\omega(c_i) \quad \text{(by Lemma 4.1)}\\
&= m\left(\sum_{c_i}\omega(c_i)t^{W(c_i)} - \sum_{c_i}\omega(c_i)\right) = m\cdot A_K(t). \qquad \square
\end{aligned}$$

Corollary 4.2. *Let K be a virtual knot of period m with a factor knot K_*. Then the affine index polynomial $A_K(t)$ of K is given as*

$$A_K(t) = m \cdot A_{K_*}(t).$$

Example 4.3. (1) The Affine index polynomial $A_{K_{4.27}}(t)$ of $K_{4.27}$ is $t^{-1}+2-t-2t^3$. Since the affine index polynomial takes values in a Laurent polynomial ring with coefficients in $\mathbb{Z}$, $K_{4.27}$ does not admit any diagram with periodic structure by Corollary 4.2.

(2) It is known that the virtual knot $K_{4.73}$ has 2-period structure, see [8]. By a direct calculation, one can see that the affine index polynomial $A_{K_{4.73}}(t)$ of $K_{4.73}$ is $2(-t^{-1}-t+2)$. The Gauss diagram $G(K_{4.73})$ can be given as the



$\mathrm{K}_{4.27}$

Fig. 14.

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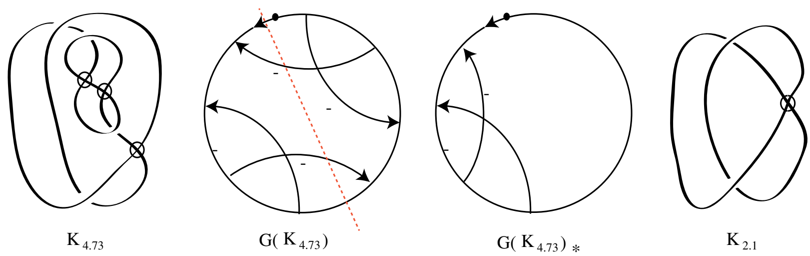

Fig. 15.

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second picture of Fig. 15 which admits 2-periodic structure. The factor Gauss diagram $G(K_{4.73})_*$ can be given as the third picture of Fig. 15 that corresponds to the virtual knot of the last picture of Fig. 15. Furthermore, one can see that the last picture is a diagram of $K_{2.1}$ and the affine index polynomial of $K_{2.1}$ is $-t^{-1} - t + 2$.

4.2. *The writhe polynomial of a periodic virtual knot*

Without loss of generality, we may assume that every chord in a Gauss diagram is a straight line segment. A chord may meet other chords in the Gauss diagram. A chord is said to be *odd* if it meets an odd number of chords in the Gauss diagram; otherwise, it is said to be *even*, see [9]. The parity of a crossing in a virtual knot diagram can be defined as the parity of the corresponding chord. Let $\mathrm{Odd}(K)$ and $\omega(c)$ denote the set of all odd crossings of K and the writhe number of a crossing c of K, respectively. The *odd writhe* $J(K)$ of a virtual knot K is defined as

$$J(K) = \sum_{c_i \in \mathrm{Odd}(K)} \omega(c_i).$$

If K is a classical knot, then $\mathrm{Odd}(K) = \phi$ so that $J(K) = 0$. It is an invariant of virtual knots, even though the writhe is not an invariant of classical knots.

Let K be a virtual knot diagram and $\phi : E(K) \to \mathbb{Z}_m$ a voltage assignment. Suppose that the net voltage $\sum_{e \in E(K)} \phi(e)$ is relative prime to m so that $K \times_\phi \mathbb{Z}_m$ is an m-periodic knot diagram. Let c be a crossing of K. Then the fiber of c consists of the crossings $(c, 0), (c, 1), \ldots, (c, m-1)$ of $K \times_\phi \mathbb{Z}_m$.

Lemma 4.3. *If a crossing c is an odd crossing (respectively, an even crossing) of K, then the crossings $(c, 0), (c, 1), \ldots, (c, m-1)$ which are in the fiber of c are also odd crossings (respectively, even crossings) of $K \times_\phi \mathbb{Z}_m$.*

Proof. From Theorem 3.1, the Gauss diagram $G(K) \times_\varphi \mathbb{Z}_m$ is the Gauss diagram of $K \times_\phi \mathbb{Z}_m$ whose factor is $G(K)$. Suppose that c is an odd crossing of K. Then the corresponding chord c' of $G(K)$ is also odd. That is, there are odd vertices, say a, from c'^o to c'^u on the Gauss diagram $G(K)$ according to the orientation of the circle of $G(K)$. Since the number of vertices of $G(K)$ is $2|V(K)|$, from the construction of $G(K) \times_\varphi \mathbb{Z}_m$, the number of vertices from the vertex (c'^o, g) to the vertex (c'^u, g) on the Gauss diagram $G(K) \times_\varphi \mathbb{Z}_m$ according to the orientation of the circle of $G(K) \times_\varphi \mathbb{Z}_m$ is $a + 2i|V(K)|$ for some $i \in \mathbb{Z}_m$, where $|V(K)|$ is the number of crossings of K. Since a is odd, the chord (c, g) is odd.

Suppose that c is an even crossing of K. The proof is similar to the previous case. The proof is finished. □

Theorem 4.4. *The odd writhe $J(K \times_\phi \mathbb{Z}_m)$ of $K \times_\phi \mathbb{Z}_m$ is given as*

$$J(K \times_\phi \mathbb{Z}_m) = m \cdot J(K).$$

Proof. Suppose that $\mathrm{Odd}(K) = \{c_1, c_2, \dots, c_n\}$. By Lemma 4.3, $\mathrm{Odd}(K \times_\phi \mathbb{Z}_m) = \{(c_i, g) \,|\, c_i \in V(K), g \in \mathbb{Z}_m\}$.

$$\begin{aligned} J(K \times_\phi \mathbb{Z}_m) &= \sum_{(c_i,g)} \omega((c_i, g)) \\ &= \sum_{(c_1,g)} \omega((c_1, g)) + \sum_{(c_2,g)} \omega((c_2, g)) + \cdots + \sum_{(c_n,g)} \omega((c_n, g)) \\ &= m \sum_{c_1} \omega(c_1) + m \sum_{c_2} \omega(c_2) + \cdots + m \sum_{c_n} \omega(c_n) \quad \text{(by Lemma 4.3)} \\ &= m \cdot J(K). \end{aligned}$$

□

Corollary 4.4. *Let K be a virtual knot of period m with factor knot K_*. Then the odd writhe $J(K)$ of a virtual knot K is given as*

$$J(K) = m \cdot J(K_*).$$

In [2], Cheng defined the odd writhe polynomial that is defined as follows. Let K be a virtual knot diagram and $G(K)$ the associated Gauss diagram of K. First, choose any edge e on the circle of $G(K)$ and assign an integer x to e arbitrarily. When one move along the orientation of the circle, the terminal point of e meets a chord c of $G(K)$. There are four cases according to the sign on c and to the orientation on c, as in Fig. 16. To the edge f on the circle meeting e at the chord c, assign $x + 1$ or $x - 1$ according to the rule in Fig. 16. It is known that one can assign an integer to each edge on the circle of $G(K)$ by repeating this process. The labeling rule around a chord is given in Fig. 17.

For each chord c_i, $N(c_i)$ is defined by

$$N(c_i) = \begin{cases} x - y, & \omega(c_i) = +1, \\ z - w, & \omega(c_i) = -1. \end{cases}$$

Then the *odd writhe polynomial* of K is defined as

$$f_K(t) = \sum_{c_i \in \mathrm{Odd}(K)} \omega(c_i) t^{N(c_i)}.$$

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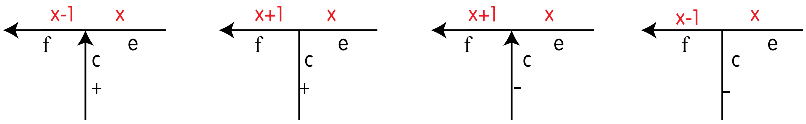

Fig. 16.

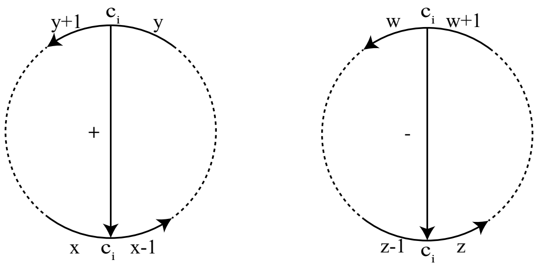

Fig. 17.

The odd writhe polynomial $f_K(t)$ is a virtual knot invariant and it can distinguish some virtual knots from its inverse and mirror image, see [2]. In [3], Cheng and Gao showed that $N(c_i) = W(c_i) + 1$.

Theorem 4.5. *The odd writhe polynomial $f_{K\times_\phi \mathbb{Z}_m}(t)$ of $K \times_\phi \mathbb{Z}_m$ is given as*

$$f_{K\times_\phi \mathbb{Z}_m}(t) = m \cdot f_K(t).$$

Proof. Suppose that $\mathrm{Odd}(K) = \{c_1, c_2, \ldots, c_n\}$. By Lemma 4.3, $\mathrm{Odd}(K \times_\phi \mathbb{Z}_m) = \{(c_i, g) \,|\, c_i \in \mathrm{Odd}(K), g \in \mathbb{Z}_m\}$.

$$\begin{aligned}
f_{K\times_\phi \mathbb{Z}_m}(t) &= \sum_{(c_i,g)} \omega((c_i,g))t^{N((c_i,g))} \\
&= \sum_{(c_1,g)} \omega((c_1,g))t^{N((c_1,g))} + \sum_{(c_2,g)} \omega((c_2,g))t^{N((c_2,g))} \\
&\quad + \cdots + \sum_{(c_n,g)} \omega((c_n,g))t^{N((c_n,g))} \\
&= \sum_{(c_1,g)} \omega((c_1,g))t^{W((c_1,g))+1} + \sum_{(c_2,g)} \omega((c_2,g))t^{W((c_2,g))+1} \\
&\quad + \cdots + \sum_{(c_n,g)} \omega((c_n,g))t^{W((c_n,g))+1} \\
&= m\sum_{c_1} \omega(c_1)t^{W(c_1)+1} + m\sum_{c_2} \omega(c_2)t^{W(c_2)+1}
\end{aligned}$$

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$$+\cdots+m\sum_{c_n}\omega(c_n)t^{W(c_n)+1} \quad \text{(by Lemma 4.1)}$$

$$= m\sum_{c_1}\omega(c_1)t^{N(c_1)} + m\sum_{c_2}\omega(c_2)t^{N(c_1)}$$

$$+\cdots+m\sum_{(c_n)}\omega(c_n)t^{N(c_n)} = m\cdot f_K(t). \qquad \square$$

Corollary 4.5. *Let K be a virtual knot of period m with factor knot K_*. Then the odd writhe polynomial $f_K(t)$ of a virtual knot K is given as*

$$f_K(t) = m\cdot f_{K_*}(t).$$

But if a virtual knot diagram has no odd crossing, then the odd writhe polynomial is trivial, so that Cheng and Gao [3] defined the writhe polynomial. Given a Gauss diagram, consider a chord c of it. Without loss of generality, we may assume that the direction of the chord c is directed from top to bottom as seen in Fig. 18. The chord c may meet other chords in the Gauss diagram. Let r_+(respectively, r_-) denote the number of positive (respectively, negative) chords which meet c whose direction from left to right, let l_+(respectively, l_-) denote the number of positive (respectively, negative) chords which meet c whose direction from right to left as seen in Fig. 18. The *index* of c is defined as

$$\mathrm{Ind}(c) = r_+ - r_- - l_+ + l_-.$$

In order to define the writhe polynomial, they used a sequence of parities which is defined by Manturov in [9]. For a given crossing c in K, define a sequence of parities as follows: c is

$$\begin{cases} \text{odd}, & \text{if } \mathrm{Ind}(c) = 2^k \mod 2^{k+1}; \\ \text{even}, & \text{if } \mathrm{Ind}(c) = 0 \mod 2^{k+1}. \end{cases}$$

Define a sequence of polynomials as follows.

$$f_k(t) = \sum_{c_i\in \mathrm{Odd}_K(K)} w(c_i)t^{N(c_i)},$$

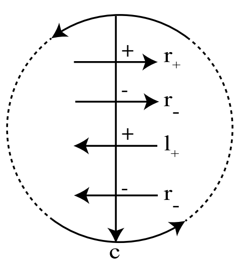

Fig. 18.

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where $\mathrm{Odd}_k(t)$ denotes the set of all odd crossings of K according to the above parity. Hence, The *writhe polynomial* of a virtual knot K was defined by

$$W_K(t) = \sum_{k=0}^{\infty} f_k(t).$$

Proposition 4.6 ([3]). *Given a virtual knot diagram K, the writhe polynomial of K is given as*

$$W_K(t) = (A_K(t)) + \sum_{\mathrm{Ind}(c_i)\neq 0} \omega(c_i))t,$$

where $A_K(t)$ is the affine index polynomial of K.

Proposition 4.7 ([3]). *For a crossing c_i in K, $N(c_i) = \mathrm{Ind}(c_i) + 1$.*

Lemma 4.8. *For a crossing c_i in K, $N(c_i) = N((c_i, g))$ for all $g \in \mathbb{Z}_m$.*

Proof. By Lemma 4.1, $W(c_i) = W((c_i, g))$ for all $g \in \mathbb{Z}_m$. Since $N(c_i) = W(c_i)+1$, $N(c_i) = N((c_i, g))$ for all $g \in \mathbb{Z}_m$. □

Theorem 4.9. *The writhe polynomial $W_{K\times_\phi \mathbb{Z}_m}(t)$ of $K \times_\phi \mathbb{Z}_m$ is given as*

$$W_{K\times_\phi \mathbb{Z}_m}(t) = m \cdot W_K(t).$$

Proof. Suppose that the set $V(K)$ of all classical crossings of K is $\{c_1, c_2, \ldots, c_n\}$. Then the set $V(K \times_\phi \mathbb{Z}_m)$ of all classical crossings of $K \times_\phi \mathbb{Z}_m$ is $\{(c_i, g) \mid c_i \in V(K), g \in \mathbb{Z}_m\}$.

$$\begin{aligned}
W_{K\times_\phi \mathbb{Z}_m}(t) &= \left(A_{K\times_\phi \mathbb{Z}_m}(t) + \sum_{\mathrm{Ind}((c_i,g))\neq 0} \omega((c_i, g))\right) t \quad \text{(by Proposition 4.6)}\\
&= \left(mA_K(t) + \sum_{\mathrm{Ind}((c_i,g))\neq 0} \omega((c_i, g))\right) t \quad \text{(by Theorem 4.2)}\\
&= \left(mA_K(t) + \sum_{N((c_i,g))\neq 1} \omega((c_i, g))\right) t \quad \text{(by Proposition 4.7)}\\
&= \left(mA_K(t) + m\sum_{N(c_i)\neq 1} \omega(c_i)\right) t \quad \text{(by Lemma 4.8)}\\
&= m\left(A_K(t) + \sum_{\mathrm{Ind}(c_i)\neq 0} \omega(c_i)\right) t \quad \text{(by Proposition 4.7)}\\
&= m \cdot W_K(t) \quad \text{(by Proposition 4.6).}
\end{aligned}$$

□

Corollary 4.6. *Let K be a virtual knot of period m with factor knot K_*. Then the writhe polynomial $W_K(t)$ of a virtual knot K is given as*

$$W_K(t) = m \cdot W_{K_*}(t).$$

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